An investigative task incorporating computerized technology with conserved property and generalization

Ruti Segal *Moshe Stupel*
rutisegal@gmail.com stupel@bezeqint.net
Shaanan Religious College of Education, Haifa, Israel

Abstract

The use of dynamic geometry software in exploring and investigating geometrical tasks is important for teaching mathematics, especially when paired with appropriate investigative tasks that can inspire while illustrating various properties and hypotheses. In this paper, we exemplify this by examining a specific investigative task through the use of illustrative software, which was tested among pre- and in-service mathematics teachers. The teachers' response to the use of the software in solving the task and developing a proof for a hypothesis is observed. We emphasize the contribution a stimulating investigative task has in improving the quality of teaching as part of a micro-investigation, and study the students' willingness to utilize the technological tools.

The particular task is presented in detail. It entailed finding the relative area of a quadrilateral formed at the vertex of a triangle by pencils issuing from the other two vertices. The relative areas were found to be conserved even as the side lengths of the triangle changed and the mathematical explanation is given. However, the task also led to a surprising discovery in that all *the relative areas of any polygon within the triangle were conserved when the line pencils cut segments of different lengths of the opposite sides. A mathematical explanation is given for this case as well. Based on their experience, students were asked to provide a proof for particular cases of number of lines in the pencil, followed by a generalized proof for any number of lines.*

1. Introduction

Research in education is constantly seeking methods to improve the quality of teaching. One issue often focused upon is the integration of technology. Dynamic geometry software (DGS) is a technological tool that allows the representation of mathematical equations and the construction of mathematical objects in a manner that provides constant feedback to the user [1,2]. By using appropriate technology to combine the process of construction and proof, dynamic software accelerates activities of discovery and investigation [3]. It provides an opportunity for students to consider not only the "Why…?" but also the "What if…?" and "What if not [4,5]. With dynamic software, learners are able to discover mathematical phenomena and models firsthand, leading to a better appreciation of the different representations and graphical relationships that mathematical concepts offer [6]. By using technological tools, mathematical concepts and relations are more easily understood and perceived, allowing learners easier access to high-level mathematics [7]

DGS helps students solve problems by learning from examples. The ability of computer technology to rapidly generate numerous, diverse examples, to store and restore moves, and to provide qualitative feedback provides the learner with valuable information concerning the mathematical concept(s) under study, information that constitutes a basis both for generalizations and for hypotheses which require proof [8,9,10,11,12]. Students are able to deduce the essential procedures from the examples by following the critical properties of the concept, internalize the concept, and then subsequently apply that concept to the solution of new problems.

In their paper Iranzo & Fortuny, [13] presented students' investigative activities in a DGS environment. Their research showed that the software allowed students to identify dependent and independent objects while manipulating them. It also allowed students to construe critical properties that aided them in the transition to a deductive proof, particularly through the identification of equal areas of triangles and equal differences between the areas. In addition, their research also pointed out the significant role that the teacher's pedagogic knowledge has in this situation: the teacher's remarks were important in guiding the students in their investigations, showing them how to make rational use of the dynamic software to identify false assumptions, and to ultimately discover the deductive proof.

However, the choice of task is paramount. To encourage such an approach requires the formulation and construction of appropriate tasks that will encourage students to formulate hypotheses as they focus on the relationships between geometrical objects and understand the reasons for the results obtained therefrom.

2. Purpose and method

With the above in mind, the authors set out to study how DGS software was perceived by 45 pre- and in-service teachers of mathematics who were attending two academic colleges of education. Most of the participants were pre-service teachers specializing in teaching high-school mathematics, but 15-25% of the students at each institution were in-service teachers completing their pedagogic education, with a focus on mathematics. The study spanned two semesters.

In the first semester, participants took part in a course entitled “Integration of a technological tool in the teaching of mathematics.” Here, they were introduced to different computerized technological tools and were taught how to use them to solve routine mathematical tasks with which they were already familiar. In the second semester, they were given specially designed investigative mathematical tasks and were asked to investigate them both from the regular mathematical aspect and with DGS — in this case, Geogebra Software. Our goal was to determine whether using DGS would lead them to discover properties that permitted the generalization of some mathematical hypotheses. To this end, the students were subsequently asked to mathematically prove the conclusions obtained from the investigative process with the DGS.

To assess the students’ impressions regarding their experience with DGS, discussions and opinions expressed during activities and follow-up interviews after the course were documented, and a follow-up questionnaire was given.

3. Original investigative task

We created a preliminary task, which asked to calculate the relative area of the quadrilateral formed at the vertex of any triangle where pencils of straight lines dividing the opposite side into equal segments are drawn from the other two vertices. For example: Calculate the relative area of quadrilateral BHKD (with respect to the area of triangle ABC) where a pencil of three straight lines is drawn from vertex A, and a pencil of two lines is drawn from vertex C (Figure 1).

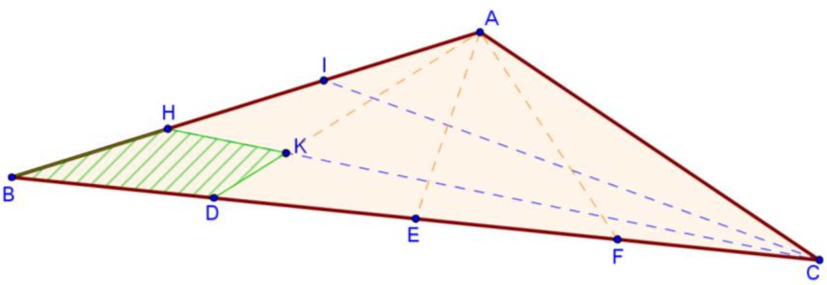

Fig. 1. The quadrilateral formed at the vertex of the triangle by the intersection of 2- and 3-line pencils.

Initially, the students were given a handout with the diagram in Figure 1 and were asked to surmise whether the ratio between the area of the quadrilateral BHKD and the triangle ABC was a constant number even when the length of the sides of the triangle is changed. Following that, they were allowed to use DGS to explore the question. The assumption was that by using DGS to investigate the problem, students would be able to construct a variety of cases that comply with the conditions, thereby elucidating how changes in the division of the triangle affects the ratio of the areas.

In order to illustrate changes, a dynamic applet was required. Some participants were able to construct suitable applets by themselves. Others received prepared applets for the purpose of the investigative activity in which the vertices of triangles with 2- and 3-line pencils could be dragged, thereby varying the lengths of its sides and changing its area and that of the quadrilateral in question (see link 1). At each stage, the areas of the quadrilateral at vertex B (labeled poly1) and the triangle ABC (poly2), and the ratio between them (p) appear on the screen.

(Note: Throughout the paper, direct links to the applets provided, in which one can view the respective constructions carried out using free Geogebra software. In order to open these links, Geogebra must be downloaded and installed on the computer.)

Link 1: The relative area of the quadrilateral created by straight-line pencils in a triangle: 2- and 3-line pencils.

A second applet was also prepared, in which the numbers of straight lines in the pencils that issue from vertices A and C (m and n, respectively) could be changed (see link 2).

Link 2: The relative area of the quadrilateral created by straight-line pencils in a triangle: numbers of lines in pencils may vary.

For the first stage, we constructed examples of particular cases. In the case of m=1, n=1, which corresponds to the case of two medians in a triangle, the ratio of the areas was always 1/3, independent of the lengths of the sides of the triangle.

In the case of m=4, n=3 (BC is divided into four equal parts and AB is divided into three equal parts) the ratio of the areas was constant at 0.13, regardless of changes in the lengths of the sides (see Figure 2).

In a similar manner, other constructions can be made. For example Figures 3-a and 3-b show triangles whose sides AB and BC are divided into five and four equal parts, respectively. One can see that the ratio, p, of the areas of quadrilateral BKGF (poly3) and triangle ABC (poly4) is equal to 0.08 even when one of the vertices of the triangle has been dragged and the length of the sides of the triangle is changed.

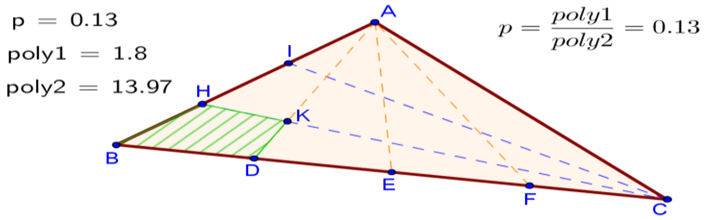

Fig. 2. The quadrilateral formed at the vertex of the triangle by the intersection of 2- and 3-line pencils, showing accompanying notation of the relative areas and ratios in question.

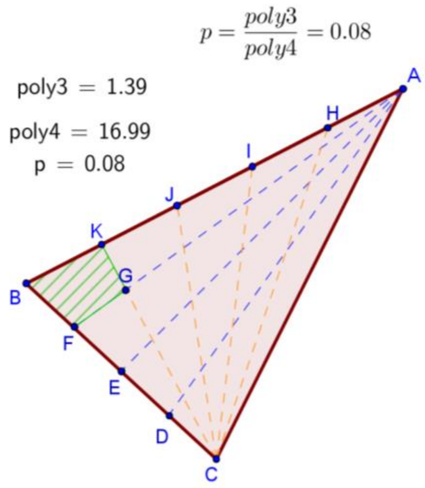

Fig. 3-a. Conservation of the relative area of the quadrilateral even though the area of the triangle changes.

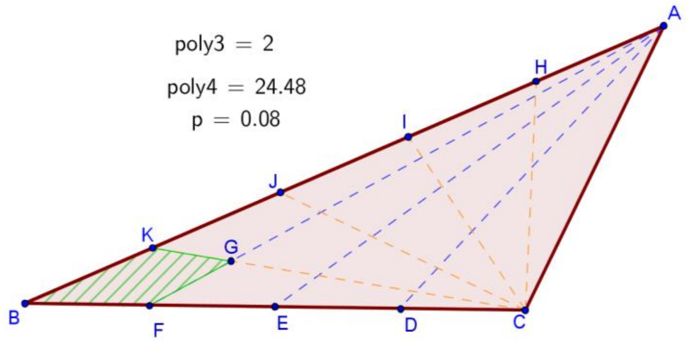

Fig. 3-b. Conservation of the relative area of the quadrilateral even though the area of the triangle changes.

3.1 Mathematical calculation of the relative area of a formed quadrilateral

It is known that the median of a triangle bisects it into two triangles with equal areas; when the three medians are drawn, the result is six triangles with equal areas, as shown in Figure 4. The equality of the areas of the six triangles can be used to prove the theorem that the medians of the triangle bisect each other in a ratio of 2:1. (This theorem has, of course, been proven by other methods that do not rely on the formed equal areas.) Denoting the point of intersection of the medians by M, we find that for any triangle, the area of quadrilateral BFMD is one third of the area of the given triangle ABC, irrespective of the lengths of its sides.

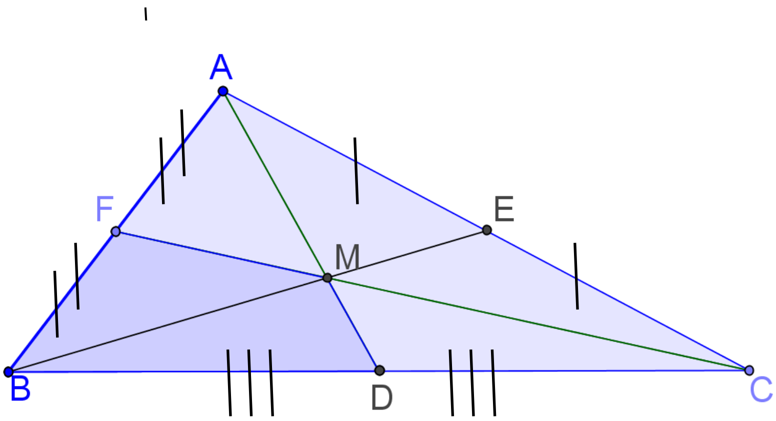

Fig. 4.The area of a quadrilateral obtained by drawing medians.

The interesting natural follow-up is an investigation of what happens to the relative areas when straight-line pencils issue from a vertex to intersect the opposite side and divide it into equal segments. Will the area of the formed quadrilateral by vertex B also cut a constant part of the area of the triangle irrespective of the side lengths of the triangle?

We first calculated the area of the quadrilateral formed for the case when three straight lines issue from vertex A (m = 3), and two straight lines issue from vertex C (n = 2), as shown in Figure 5. The area of quadrilateral BHKD must be expressed using the area of triangle ABC (S).

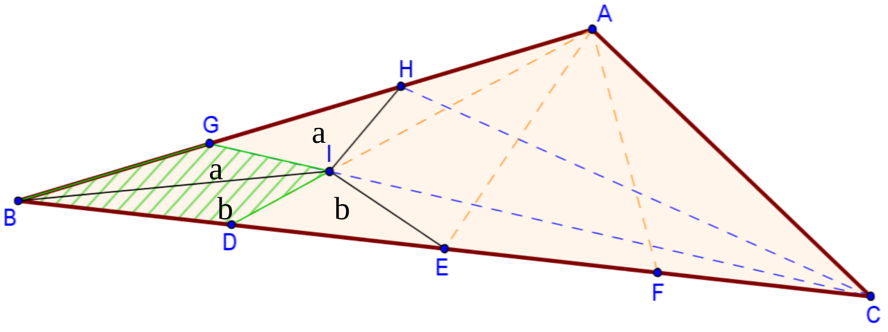

Fig.5. Calculating the relative area of a quadrilateral formed by the intersection of 2- and 3-line pencils.

We denote: $S_{\Delta BGI} = a$, hence it follows that $S_{\Delta IGH} = S_{\Delta IHA} = a$.

We denote: $S_{\Delta BDI} = b$, hence it follows that $S_{\Delta DIE} = b$, $S_{\Delta EIC} = 2b$.

From this, we obtain a set of two equations:

$$3a + b = \frac{s}{4}$$

$$a + 4b = \frac{s}{3}$$

Solving the equations gives an expression for *a* and for *b* as parts of S.

Hence we obtain: $S_{BGID} = \frac{17}{132} S$.

3.2 The general case

Side BC divided into m equal parts, and side AB divided into n equal parts. The notation of the areas of the triangles is similar to the specific case, above.

Hence, follow the equations:

$$na + b = \frac{s}{m}$$

$$a + mb = \frac{s}{n}$$

Solving the set of equations we obtain: $S_{BHKD} = \frac{(2mn - m - n)S}{mn(mn-1)}$.

Thus, for given values of m and n, the ratio between the areas of formed quadrilateral BHKD and given triangle ABC remains constant and independent on the side lengths of the triangle. For the particular case of m = 1, n= 1, $S_{BHKD} = \frac{1}{3} S_{\Delta ABC}$.

4. Expanded investigative task

After the students had worked out the general solution, the researchers noted that students further made use of DGS to explore how changes in parameters affected the relative areas of *other* polygons adjacent to the triangle's sides. For example, one student was intrigued to note that no matter how vertex B of the triangle was dragged (see Figures 6-a and 6-b), the software indicated that the ratios (p1) of the areas of quadrilateral BDGI (poly5) and triangle ABC (poly2) remained the same. It seemed that use of the dynamic tool did indeed accelerate activities of discovery and investigation [3] by allowing participants an easy way to experiment with the parameters. Here to, they were surprised to see the ratios remain constant throughout the manipulations (see Link 3).

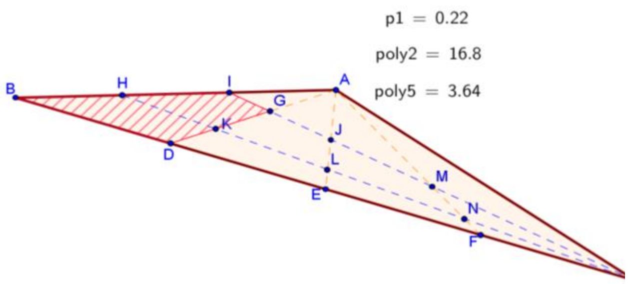

Fig. 6-a. Demonstration of the conservation of the relative area of adjacent corner quadrilaterals even though the area of the triangle is changed.

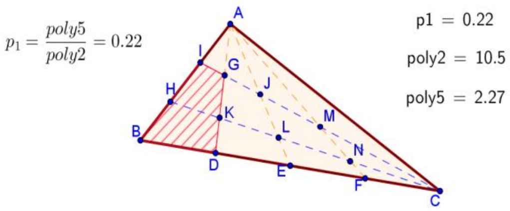

Fig. 6-b. Demonstration of the conservation of the relative area of adjacent corner quadrilaterals even though the area of the triangle is changed.

Link 3: The relative area of contiguous regions created by straight-line pencils and adjacent to the side of a triangle.

This discovery prompted us to probe further and check if this phenomenon also occurs when the polygon is *not* directly adjacent to the sides of the triangle. Again, the DGS allowed us to easily check particular cases, leading to the hypothesis that the phenomenon is indeed true, as described in Figures 7-a and 7-b (see Link 4).

Link 4: The relative area of adjacent regions inside the triangle created by straight-line pencils in a triangle.

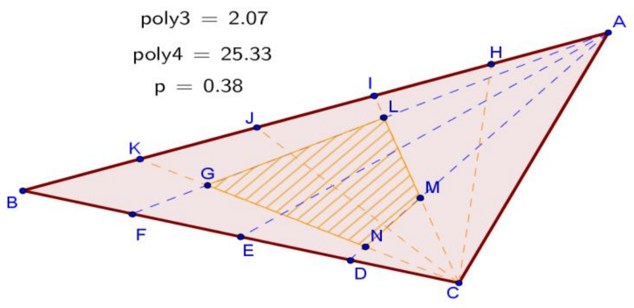

Fig. 7-a. Demonstration of the conservation of the relative area of an internal polygon.

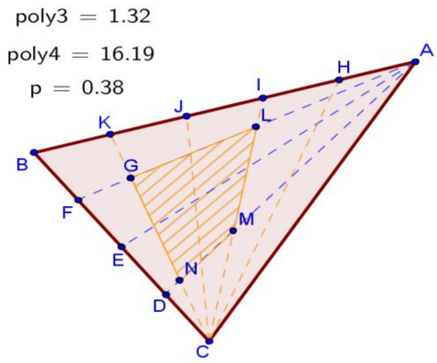

Fig. 7-b. Demonstration of the conservation of the relative area of an internal polygon.

While the researchers were aware of the conservation of the ratio of areas when applied to quadrilaterals positioned at the vertex of a triangle (see proof, above), the additional, unexpected results obtained from the use of the Geogebra software concerning conservation of relative area for any shape along the sides or within the triangle were surprising and demanded a mathematical explanation.

4.1. General proof for the conservation of the relative area of any quadrilateral formed at vertex B by means of two straight lines that divide the sides AB and BC in any ratio.

In triangle ABC with area S, we draw a straight line, AD, which bisects side BC into two segments whose lengths are x_1 and x_2, and a straight line, CE, that bisects side BE into two segments whose lengths are y_1 and y_2. AD and CE intersect at point F, as shown in Figure 8.

We wish to express the area of quadrilateral BEFD in terms of x_1, x_2, y_1 and y_2.

We draw the diagonal BF of the quadrilateral and denote the following areas:

$$S_1 = S_{\Delta BFD}$$

$$S_2 = S_{\Delta DFC}$$

$$A_1 = S_{\Delta BFE}$$

$$A_2 = S_{\Delta EFA}$$

Also:

$$\frac{S_1}{S_2} = \frac{x_1}{x_2} \Rightarrow S_2 = \frac{x_2}{x_1} S_1$$

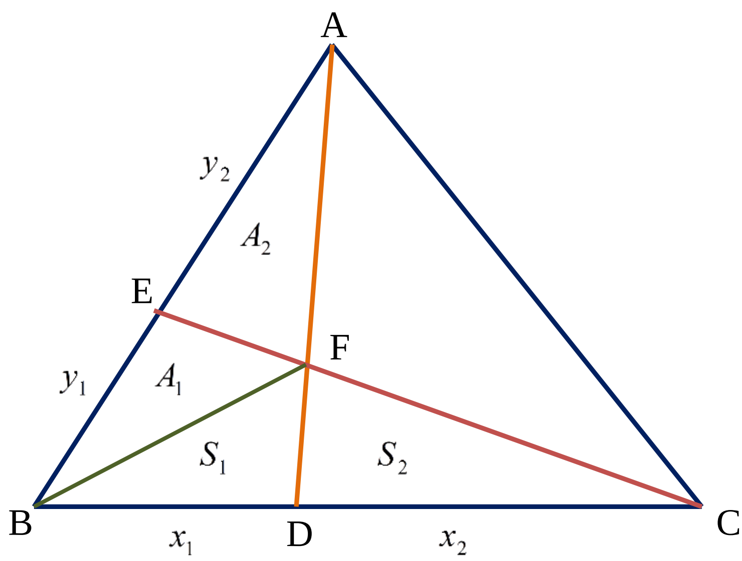

Fig. 8. The notation for calculating the relative area of a quadrilateral formed from the intersection of pencils that do not cut segments of equal length from the sides of the triangle.

$$\frac{A_1}{A_2} = \frac{y_1}{y_2} \Rightarrow A_2 = \frac{y_2}{y_1} A_1$$

$$A_1 + S_1 + S_2 = \frac{y_1}{y_1 + y_2} S$$

$$A_1 + A_2 + S_1 = \frac{x_1}{x_1 + x_2} S \ .$$

By substituting S_2 and A_2, we obtain:

$$A_1 + \left(1 + \frac{x_2}{x_1}\right) S_1 = \frac{S}{1 + \frac{y_2}{y_1}} \quad \text{and} \quad \left(1 + \frac{y_2}{y_1}\right) A_1 + S_1 = \frac{S}{1 + \frac{x_2}{x_1}} \ .$$

We denote: $t = \frac{x_2}{x_1}$ and $p = \frac{y_2}{y_1}$.

From here, we obtain a system of first-order equations for S_1 and A_1 as a function of S, *t* and *p*.

For a given *p* and *t*, we obtain the area of the quadrilateral BEFD (the sum of A_1 and S_1) as $S_{BEFD} = k \cdot S$, where the magnitude of *k* is determined by *p* and *t*, which are the ratios of segment lengths x_1 and x_2, and y_1 and y_2, respectively. This implies that *k* is constant and is the ratio between the areas of quadrilateral BEFD and triangle ABC.

The proof that shows that the relative area of a quadrilateral obtained at vertex B is constant, and independent of the side lengths of triangle ABC, (it depends only on the ratio of the lengths cut by the straight lines off the sides) allows one to prove conservation of the relative area of regions included within the web of intersecting lines. To this end, we may draw any three intersecting lines from the vertex A and any two intersecting lines from the vertex C, as shown on Figure 9.

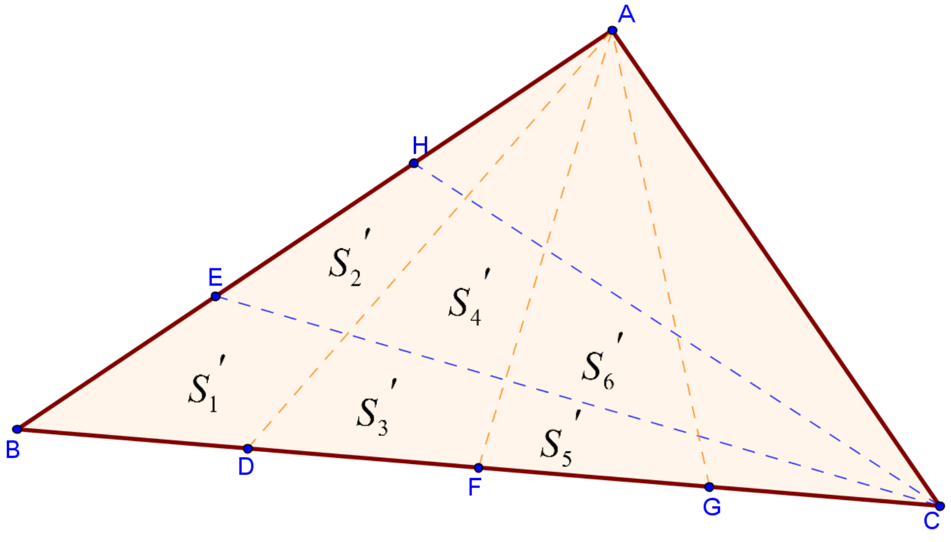

Fig. 9. The notations for calculating the relative area of each quadrilateral formed by the intersection of a pencil of lines with the sides of the triangle

The relative areas of each quadrilateral obtained from the intersection of the straight lines are denoted by $S_1', S_2', S_3', S_4', S_5', S_6'$.

We have proved that S_1', the relative area of the quadrilateral at vertex B, is constant.

Similarly, the relative area of the quadrilateral formed by the straight lines AD and CH, $S_1' + S_2'$, is constant based on the general proof, and since S_1' is constant, S_2' must also be constant, and in the same manner the relative area of the quadrilateral formed by the straight lines AD and CE, $S_1' + S_3'$, is constant, as is S_3'.

In the same manner, we can prove that the relative area of the quadrilateral formed by the straight lines AF and CH, $S_1' + S_2' + S_3' + S_4'$, is constant, and since we proved that S_1', S_2', S_3' are constant, S_4' is also constant.

In a similar fashion, it can be proven by recursion from the vertex B and into the triangle that the relative area of *any* quadrilateral (or triangle) is constant and does not depend on the side lengths of the triangle. Thus, by combining several quadrilaterals or triangles in the web (even if not contiguous), their relative cumulative area shall be preserved.

4.2. Proof for the conservation of the relative area of any quadrilateral formed at vertex B by means of straight-line pencils that divide sides AB and BC equally.

Because the above general proof for the conservation of the relative area of the quadrilateral at vertex B stipulates that the two straight lines cut segments of the opposite sides in ratios $x_1 : x_2$ and $y_1 : y_2$ (where x_1, x_2 and y_1, y_2 are not necessarily whole numbers) and does *not* rely on the theorem regarding the bisection of the area of triangle by a median into two triangles with equal areas, it also constitutes proof for the case in which the pencils cut segments with equal lengths on the opposite sides, with the single difference being that in this case the ratio of the side lengths cut of the opposite sides is a quotient of two whole numbers.

5. Results

5.1 General results

In the first stage, students were presented with a handout of Figure 1 and asked to surmise whether the ratio between the area of quadrilateral BHKD and triangle ABC would be constant even when the length of the sides of the triangle were changed. Most of them answered intuitively that it would *not* be a constant number, but could not explain why.

Following the second stage — experimentation with the DGS applets — students expressed surprise in discovering that the ratio *did* remain constant. Some students constructed additional cases using different pencils issuing from the two vertices, and were intrigued to discover that even then, the ratio between the area of the quadrilateral formed at vertex B and the area of the triangle ABC remained constant no matter how the dimensions of the triangle changed.

During the second stage, students asked to prove the hypotheses they arrived at using the DGS. They directed to start an investigative process using the particular case with two medians, and managed to prove the consistency of the ratio between the areas of quadrilateral BFMD and triangle ABC (see Figure 10) by combining two theorems: a) the medians in a triangle meet at one point that divides them in a ratio of 1:2; and b) a median divides a triangle into two triangles with equal areas.

Thus, the students concluded that $S_1+S_2+S_3$ is half the area of triangle ABC, and, based on the division ratio of the medians, they concluded that $S_1+S_2=2\cdot S_3$, and therefore S_3 is one third of half the area of the triangle, i.e., one sixth of the area of the triangle ABC. Similarly, they determined that S_2 is also a sixth of the area of the triangle ABC, leading them to conclude that the area of quadrilateral FMDB is one-third the area of triangle ABC.

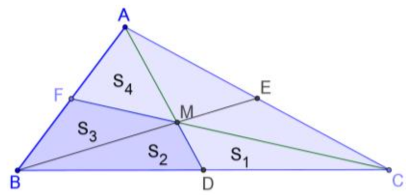

Figure 10. Proof using the medians theorem.

The transition from the proof of the particular case of the medians to other particular cases was not immediate. For example, some students were not successful in realizing that theorem b (a median divides a triangle into two triangles with equal areas) could also be used for internal triangles (e.g., that $S_3=S_4$). However, the proof was expedited once they were given the clue that when the side of a triangle is divided into equal segments by a pencil of m segments issuing from the opposite vertex, m+1 triangles with equal areas are formed. Once they understood that this property was true also for its converse, and combined it with theorem b, (a median divides a triangle into two triangles with equal areas) for the internal triangles, the theorem was easily proven. In fact, the students experienced the process of creating an infinite number of examples as a basis for hypothesizing, leading to the importance of the formal proof.

In the second stage, the students were also asked to determine if the conclusion could be generalized to other relative areas (as shown in Figures 6-a and 6-b, and 7-a and 7-b) and for different pencils. Here too, most of the students initially were doubtful. However, after using the different applets that allowed changing the numbers of lines in the pencils issuing from the vertices, and considering different relative areas within the triangle while dragging the vertices, the students discovered the constant ratios that resulted.

5.2 Qualitative survey

In order to assess how using DGS affects students' appreciation of the value of the software, students were asked to fill out a feedback form upon completion of each course (first-semester introductory DGS course using familiar problems, and second-semester DGS course using unfamiliar, intriguing problems, one of which was the one described in this paper). The following questions were to be rated on a scale of 1 (least) to 5 (most):

1. Do you believe that integration of computerized mathematical tools (DGS) contributes to the quality of mathematics instruction?

2. Will you/do you utilize computerized technological tools in your teaching?

The results are shown in tables 1 and 2.

Table 1. Feedback to questions after the familiar tasks (first semester)

Question No.	Average	No. respondents
1	2.81	38
2	2.67	40

Table 2. Feedback to questions after using unfamiliar tasks (second semester)

Question No.	Average	No. respondents
1	3.51	39
2	3.24	37

The data shows that students were more willing to incorporate DGS in the classroom after the second stage of teaching. This may be due to their exposure, in the second stage, to more complex research tasks, which led to unique discoveries that echo the revelations that students in the classroom may make upon encountering novel, intriguing ideas. The tasks given to the pre- and in-service teachers in the first course were regular, familiar tasks that offered no challenge; in the second course, they were allowed to experience, through the use of the technological tools, the thrill of discovery of unexpected, unfamiliar mathematical concepts thus strengthening the impact that DGS made on them. This led to an improvement in the students' appreciation of the potential of technological tools to improve the quality of mathematics teaching by understanding how it not only allows students to better visualize and cope with difficult problems, but to personally experience the discovery of unique ideas. This was translated into an increased willingness to incorporate DGS into their classroom.

5.3 Verbal feedback

The appreciation for the value of DGS was also expressed in the students' verbal feedback regarding their opinion of the courses. In general, students were entranced by the mathematical discoveries they made during the investigative activity in stage two, and were impressed by the power that the technological tool had as a significant partner in the investigative process. They also expressed satisfaction with the relative ease in which suitable applets could be constructed to describe the relative areas. This supported the power that DGS has, both as an aid (for learners) to more deeply understand the complex and dynamic mathematical concepts and phenomena, and as a technological enhancement (for teachers) to include interesting investigative activities in the class environment.

Following are some comments from pre- and in-service teachers:

> "The computerized tool exhibited the property of conservation of the ratio of the areas visually and clearly. It was quite astonishing."

> "Working with the computer allowed us to investigate an unfamiliar phenomenon, to change the relative areas and to notice immediately if the property is conserved. Once we had a clear visualization of the phenomenon, we could proceed to try to understand why this is always true."

> "The work with the dynamic software allowed us to discover surprising properties that can be generalized. This is much more difficult to visualize in a textbook."

> "I will strive to integrate the technological tool in my lessons. The visual impact, the surprise, and the aesthetics of the task contribute to the understanding of the material to be taught."

> "The task offered an innovative view on the teaching of geometry."

6. Discussion and conclusions

In today's world, the rational use of a technological tool for the solution of mathematical problems in mathematics (particularly in geometry) is a real requirement for pre-service teachers. The technological environment grants access to a novel pedagogic environment that effectively combines dynamic tools for hypothesizing and then developing the proof process for a problem that may be initially presented outside the technological environment. In other words, integration of the technological tool in teaching has a significant and principal role in combining the technological and deductive environments in the teaching of geometry. Technology adds a new dimension to teaching and to the learning of mathematics, and teachers should be required to have knowledge of it [2]. Pre-service teachers should be exposed to the interrelations between all the specified factors in order to obtain technology-based teaching skills, and hence develop the reasoning processes of the students in solving problems in geometry.

Our findings show that DGS can enhance the learning experience, but moreover they also emphasize the importance of exposing pre- and in-service teachers to intriguing mathematical investigative tasks using such technological tools. The more experience teachers have in using such aids, and the more they use them for interesting investigative tasks, the more importance they will give them as a means for improving their teaching skills. Introducing the experience of discovery into their training increases their appreciation for the technology.

This need for incorporating mathematical "surprises" into the teaching process was discussed by [14]. It not only improves mathematics' "image" (by exhibiting the beauty of the discipline), it also may assist in improving teaching methods so as to provide the learner more significant and experiential learning. Thus, another conclusion is that people who write pedagogic material must provide teachers with suitable investigative tasks for using DGS. By designing and presenting tasks that have the potential of surprise, mathematical and pedagogic knowledge improved and, obliquely, more challenging and experiential learning opportunities can be offered for students in the classroom. At the same time, appropriate tasks contribute significantly to the students' methodical ability to carry out research that allows them to discover various properties (such as conservation), and to their ability to generalize, which is an important part of the training process.

It should be noted that, during the study, students were allowed to construct examples that presented particular cases of the tasks involved. This also constitutes stages of discovery — including hypothesizing and identification of conserved properties — that are essential stages on the road to formal proof. "Exploration leads to discovery, while proof is confirmation" [15].

Other authors have touched upon the importance of discovery for the student/teacher. Stupel & Ben-Chaim [16] conceptualized work in the dynamic geometry environment that gives rise to hypotheses concerning interrelations between different objects constructed by the students as "semi-proof." They found much importance in working in a dynamic environment that allows "a quasi-empirical investigation environment" for students. They found that the process of constructing and forming a multitude of examples by dragging vertices, sides, or any other object, accelerates processes of justification, comprehension, validity and discovery among the learners. In this present paper, constructions that include the division of a triangle's sides into a variable number of equal or non-equal segments, or changing the lengths of the triangle's sides by dragging the vertices, followed by calculating and identifying the constant ratio between the area of the quadrilateral and the area of the triangle constitute a similar "semi-proof" on the road to the formal proof.

Martinovic & Manizode [2] also describe the contribution of dynamic software in constructing proofs as part of a recurring pedagogic process: reflexive pedagogy in action. Of the software's different roles they choose to accentuate is its role as a partner in the learning process and how its use accelerates processes of mathematical justification. In this case, the user relies on their mathematical knowledge in order to justify what the technology illustrates. The technology accentuates the need for a formal proof [17]. Dynamic software also constitutes a significant tool in testing the correctness of processes in geometry [18]. This process is clearly demonstrated in the integration of the task presented in the present paper's investigative activity: using the software to effect repetitive processes of constructions and calculations of different areas and ratios gave rise to a recurring pedagogic process where novel discoveries were made, thus accelerating the processes leading to a formal proof of the phenomenon in general.

Finally, the essence of the task itself is important. Not only should it give pre- and in-service teachers appropriate access to expanding their technological, pedagogic and content-based knowledge [19], it needs to provide original and intriguing means for investigating the hypothesis in question. The investigative task presented in this paper (both its original and advanced versions) proved able to offer the uniqueness required to fulfil the task.

7. References

[1] Alakoc, Z. (2003). Technological modern teaching approaches in mathematics teaching. *The Turkish Online Journal of Educational Technology,* Vol.2 (1), pp.1303-652.

[2] Martinovic, D., & Manizade, A.G. (2013). Technology as a partner in geometry classrooms. *The Electronic Journal of Mathematics and Technology,* Vol.8, n.2.

[3] Hoyles, C., & Jones, K. (1998), Proof in Dynamic Geometry Context, In: C. Mammana and V. Villani (Eds.), *Perspective on the Teaching Geometry for the 21st century*. Dordrecht: Kluwer. pp. 121-128.

[4] Brown, S. I., & Walter, M. I. (1969). What if not? *Mathematics Teaching, 46,* 38-45.

[5] Brown, S. I., & Walter, M. I. (1990). *The art of problem posing* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.

[6] Wiest, L.R. (2001). The role of computers in mathematics teaching and learning. *Computers in the Schools,* Vol. 17(1), 41-55.

[7] Hohenwarter, J., Hohenwarter,m., & Lavicza, Z. (2008). Introducing dynamic mathematics software to secondary school teachers: The case of Geogebra. *Journal of Computers in Mathematics and Science Teaching,* Vol.28 (2), pp. 135-146.

[8] Heinze, A., & von Ossietzky, C. (2002). "…Because a square is not a rectangle" - Students' knowledge of simple geometrical concepts when starting to learn proof. In A. D. Cockburn,

& E. Nardi (Eds.). *Proceedings of the 26th Conference of the international group for the Psychology of Mathematics Education*, 3, (pp. 81-88). Norwich, U.K.

[9] Holzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations - A case study. *International Journal of Computer for Mathematical Learning*, Vol.6, pp. 63-86.

[10] Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, Vol. 44, pp. 55-85.

[11] Laborde, C. (2000). Dynamic Geometry Environment as a Source of Rich Learning Contexts for the Complex Activity of Providing. *Educational Studies in Mathematics*, Vol. 44, pp. 151-161.

[12] Mariotti, M. A. (2000). Introduction to Proof: The Mediation of a Dynamic Software Environment. *Educational Studies in Mathematics*, Vol. 44, pp. 25-53.

[13] Iranzo, N., & Fortuny, J. M. (2010). The synergy of students' use of paper-and- pencil techniques and dynamic geometry software: A case study, http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg7-20-iranzo.pdf.

[14] Monshovits-Hadar, N. (1988). School mathematics theorems – an endless source of surprise. *For the Learning of Mathematics*, v. 8, n. 3.

[15] Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44, 5-23.

[16] Stupel, M., & Ben-Chaim, D. (2013). One problem, multiple solutions: How multiple proofs can connect several areas of mathematics. *Far East Journal of Mathematical Education***,** Vol. 11(2), pp. 129-161.

[17] Goos, M., Galbraith, P., Renshaw, P. & Geiger, V. (2000). Reshaping teacher and student role in technology rich classrooms, *Mathematics Education Research Journal* Vol.12 (3), pp. 303-320.

[18] Stupel, M., Oxman, V., & Sigler, A. (2014). More on geometrical constructions of a tangent to a circle with straightedge only, *The Electronic Journal of Mathematics and Technology*, Vol. 8, n.1.

[19] Koehler, M., & Mishra, P. (2009). What is technological pedagogical content knowledge (TPACK)? *Contemporary Issues in Technology and Teacher Education*, 9(1), 60-67. AACE.